The crossing numbers of $K_m \times P_n$ and $K_m \times C_n$*

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Abstract

The *crossing number* of a graph G is the minimum number of pairwise intersections of edges in a drawing of G. In this paper, we study the crossing numbers of $K_m \times P_n$ and $K_m \times C_n$.

Keywords: *Crossing number*; *Drawing*; *Complete bipartite graphs*; *Kronecker product*

1 Introduction and preliminaries

Let G be a graph, $V(G)$ the vertex set and $E(G)$ the edge set. The crossing number of G, denoted by $cr(G)$, is the smallest number of pairwise crossings of edges among all drawings of G in the plane. We use D to denote a drawing of a graph G and $\nu(D)$ the number of crossings in D. It is clear that $cr(G) \leq \nu(D)$.

Let E_1 and E_2 be two disjoint subsets of an edge set E. The number of the crossings formed by an edge in E_1 and another edge in E_2 is denoted by $\nu_D(E_1, E_2)$ in a drawing D. The number of the crossings that involve a pair of edges in E_1 is denoted by $\nu_D(E_1)$. Then $\nu(D) = \nu_D(E)$. By counting the numbers of crossings in E, we have

Lemma 1.1. $\nu_D(E_1 \cup E_2) = \nu_D(E_1) + \nu_D(E_2) + \nu_D(E_1, E_2)$.

The *Kronecker product* $G \times H$ of graphs G and H has vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{\{(a, x), (b, y)\} : \{a, b\} \in E(G) \text{ and } \{x, y\} \in E(H)\}$.

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(The product is also known as direct product, cardinal product, cross product and graph conjunction.)

Computing the crossing number of graphs is a complicated yet classical problem. And it is proved that the problem is NP-complete by Garey and Johnson [4].

In literature, the Cartesian product has been paid more attention[1, 8–10], while Kronecker product has fewer results on the crossing number[5].

In this paper, we study the crossing numbers of the Kronecker product $K_m \times P_n$ and $K_m \times C_n$. In Section 2, we give an upper bound of $cr(K_m \times P_n)$ for $n \geq 4$ and $m \geq 4$. In Section 3, we give an upper bound of $cr(K_m \times C_n)$ for $n \geq 3$ and $m \geq 4$. In Section 4, we give lower bounds of $cr(K_m \times P_n)$ and $cr(K_m \times C_n)$.

2 Upper bound of $cr(K_m \times P_n)$

Let

$$\begin{aligned} V(K_m \times P_n) = &\ \{(i,j) \mid 0 \leq i \leq m-1 \text{ and } 0 \leq j \leq n-1\}, \\ E(K_m \times P_n) = &\ \{((i_1,j),(i_2,j+1)) \mid 0 \leq i_1 \neq i_2 \leq m-1 \text{ and } 0 \leq j \leq n-2\}, \end{aligned}$$

where the first subscript is modulo m.

For $0 \leq j \leq n-2$, let

$$E^j = \ \{((i_1,j),(i_2,j+1)) \mid 0 \leq i_1 \neq i_2 \leq m-1\}.$$

Then

$$\textstyle\bigcup_{j=0}^{n-2} E^j = E(K_m \times P_n), \quad E^{j_1} \bigcap E^{j_2} = \emptyset (0 \leq j_1 \neq j_2 \leq n-2).$$

In Figure 2.1, we exhibit drawings $D_{m,4}$ of $K_m \times P_4$ in a cylinder for $m \leq 10$. A cylinder can be 'assembled' from a polygon by identifying one pair of opposite sides of a rectangle[2].

By counting the numbers of crossings in $D_{m,n}$, we have

Lemma 2.1. *For $n \geq 4$,*

$$\nu(D_{m,n}) = \begin{cases} \frac{(n-1)m(m-1)(m-2)(m-3)}{6} - \frac{m(m-3)(m^2+6m-31)}{24} & \text{for odd } m \geq 5, \\ \frac{(n-1)m(m-1)(m-2)(m-3)}{6} - \frac{m(m-4)(m^2+10m-48)}{24} & \text{for even } m \geq 4. \end{cases}$$

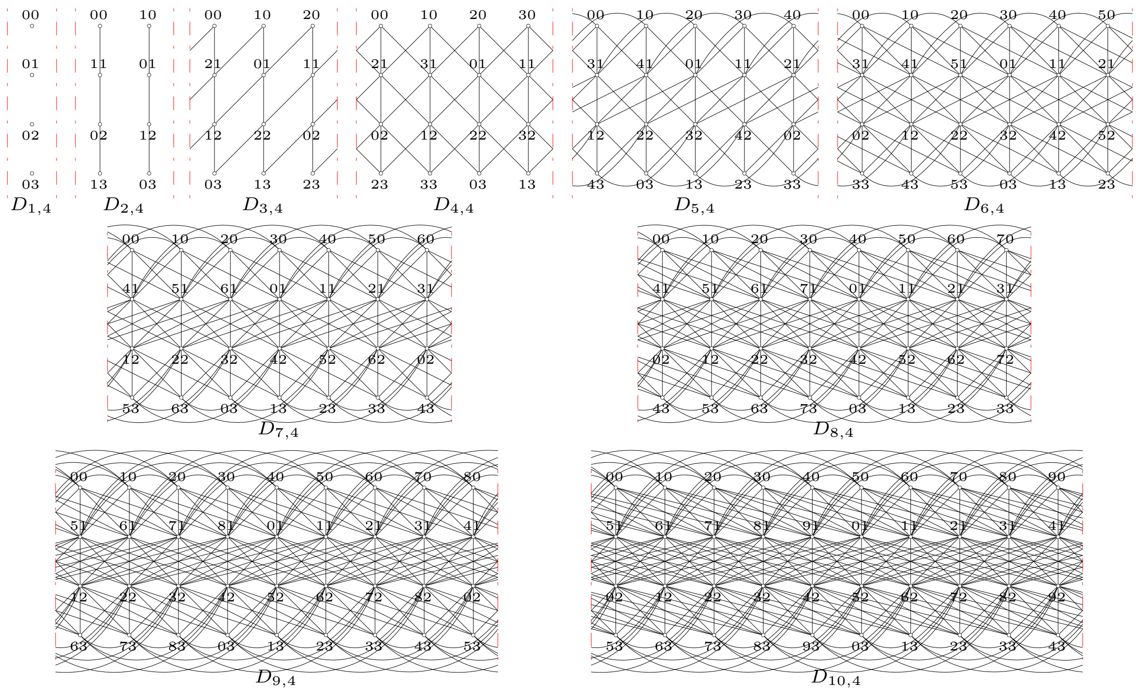

Figure 2.1: Drawings $D_{m,4}$ for $m \leq 10$

Proof. Since $\nu_{D_{m,n}}(E^{j_1}, E^{j_2}) = 0$ for $0 \leq j_1 \neq j_2 \leq n-2$, by Lemma 1.1, we have $\nu(D_{m,n}) = \sum_{j=0}^{n-2} \nu_{D_{m,n}}(E^j) = 2\nu_{D_{m,n}}(E^0) + (n-3)\nu_{D_{m,n}}(E^1)$. For $m \geq 4$,

$$\nu_{D_{m,n}}(E^1) = m\sum_{j=0}^{m-3}\sum_{i=0}^{j} i = \frac{m(m-1)(m-2)(m-3)}{6}.$$

For odd $m \geq 5$,

$$\nu_{D_{m,n}}(E^0) = \nu_{D_{m,n}}(E^1) - m\sum_{j=2}^{\frac{m-1}{2}}(\sum_{i=2}^{j} i - 1) = \frac{m(m-1)(m-2)(m-3)}{6} - \frac{m(m-3)(m^2+6m-31)}{48}.$$

For even $m \geq 4$,

$$\nu_{D_{m,n}}(E^0) = \nu_{D_{m,n}}(E^1) - m\sum_{j=3}^{\frac{m}{2}}(\sum_{i=3}^{j} i - 1) = \frac{m(m-1)(m-2)(m-3)}{6} - \frac{m(m-4)(m^2+10m-48)}{48}.$$

Hence,

$$\begin{aligned}\nu(D_{m,n}) &= 2\nu_{D_{m,n}}(E^0) + (n-3)\nu_{D_{m,n}}(E^1) \\ &= \begin{cases} \frac{(n-1)m(m-1)(m-2)(m-3)}{6} - \frac{m(m-3)(m^2+6m-31)}{24} & \text{for odd } m \geq 5, \\ \frac{(n-1)m(m-1)(m-2)(m-3)}{6} - \frac{m(m-4)(m^2+10m-48)}{24} & \text{for even } m \geq 4. \end{cases}\end{aligned}$$

□

It is easy to verify that $cr(K_m \times P_n) = 0$ for $m = 1, 2, 3$. (See Figure 2.1). For $m \geq 4$, by Lemma 2.1 we have

Theorem 2.1. *For $n \geq 4$,*

$$cr(K_m \times P_n) \quad \leq \begin{cases} \frac{(n-1)m(m-1)(m-2)(m-3)}{6} - \frac{m(m-3)(m^2+6m-31)}{24} & \textit{for odd } m \geq 5, \\ \frac{(n-1)m(m-1)(m-2)(m-3)}{6} - \frac{m(m-4)(m^2+10m-48)}{24} & \textit{for even } m \geq 4. \end{cases}$$

We will discuss the crossing number of $K_m \times P_n$ for $n = 2, 3$ in another paper.

3 Upper bound of $cr(K_m \times C_n)$

It is easy to verify that $cr(K_m \times C_n) = 0$ for $m = 1, 2$. (See Figure 3.1). For $m = 3$, $K_3 \times C_n \cong C_3 \times C_n$. By [5], $cr(K_3 \times C_3) = 3$, $cr(K_3 \times C_n) \leq 6n - 18$ for $3 < n < 9$ and $cr(K_3 \times C_n) \leq 3n$ for $n \geq 9$. In this paper, we only consider the case for $m \geq 4$.

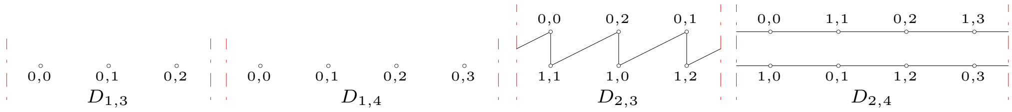

Figure 3.1: Drawings $D_{m,n}$ for $(m, n) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

Let

$$\begin{aligned} V(K_m \times C_n) = &\quad \{(i, j) \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}, \\ E(K_m \times C_n) = &\quad \{((i_1, j), (i_2, j+1)) \mid 0 \leq i_1 \neq i_2 \leq m-1, 0 \leq j \leq n-1\}, \end{aligned}$$

where the first subscript is modulo m and the second subscript is modulo n.

For $0 \leq j \leq n-1$, let

$$\begin{aligned} V^j = &\quad \{(i, j) \mid 0 \leq i \leq m-1\}, \\ E^j = &\quad \{((i_1, j), (i_2, j+1)) \mid 0 \leq i_1 \neq i_2 \leq m-1\}. \end{aligned}$$

Then

$$\textstyle\bigcup_{j=0}^{n-1} E^j = E(K_m \times C_n), \quad E^{j_1} \bigcap E^{j_2} = \emptyset (0 \leq j_1 \neq j_2 \leq n-1).$$

In Figure 3.2, we exhibit drawings $D_{m,n}$ of $K_m \times C_n$ in a cylinder for $(m, n) \in \{(4, 6), (5, 6),\ (4, 7), (5, 7), (6, 7), (7, 7)\}$. In Figure 3.3 and 3.4, we exhibit drawings $D_{m,3}$ for $4 \leq m \leq 7$ and $D_{m,5}$ for $6 \leq m \leq 11$ respectively.

In drawings $D_{m,n}$, vertices $(i_{0,0}, 0), (i_{1,0}, 1), \cdots, (i_{n-1,0}, n-1)$ $((i_{0,m-1}, 0), (i_{1,m-1}, 1), \cdots,$ $(i_{n-1,m-1}, n-1))$ are placed equidistantly on the perimeter of the top (bottom) disk, vertices $(i_{0,j}, 0), (i_{1,j}, 1), \cdots, (i_{n-1,j},\ n-1)$ are placed equidistantly on the cylinder from top

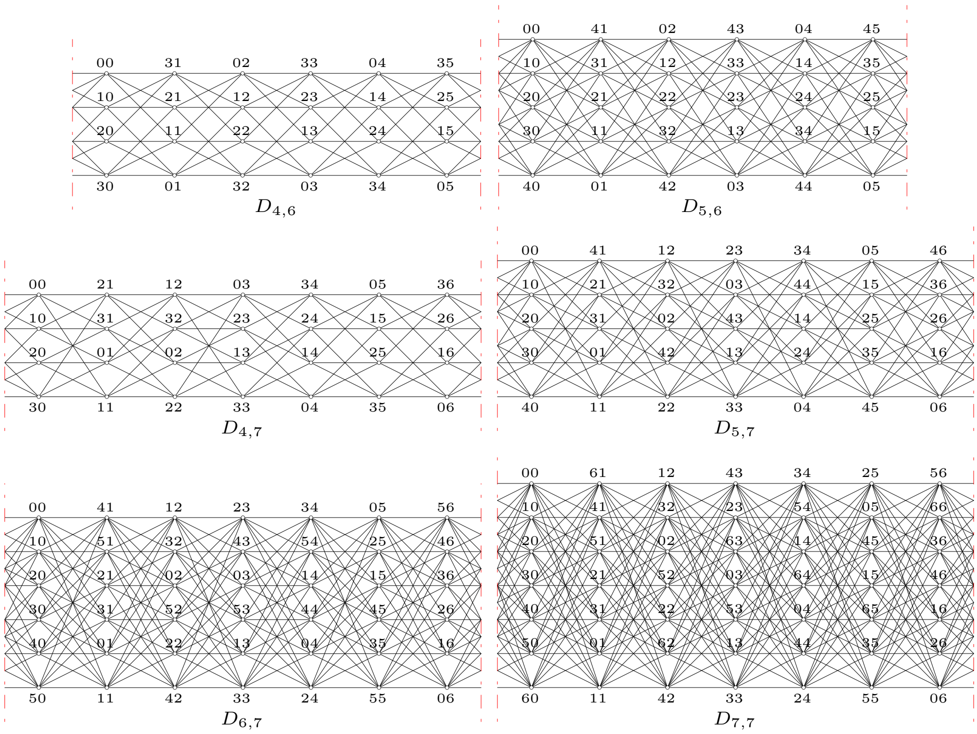

Figure 3.2: Drawings $D_{m,n}$ for $(m,n) \in \{(4,6),(5,6),(4,7),(5,7),(6,7),(7,7)\}$

to down for j from 1 to $m-2$, edges of E^j are drawn on by shortest helical curves on the cylinder. For $0 \le j \le n-1$, let $f^j = (f^j(0), f^j(1), \cdots, f^j(m-1))$ be an arrangement of $\{0,1,\cdots,m-1\}$ such that $i_{j-1,t} = i_{j,f^j(t)}$ for all $0 \le t \le m-1$, where j is modulo m. In drawings $D_{m,n}$, $i_{0,t} = t$ for $0 \le t \le m-1$.

For $m \ge 4$, let

$$
\begin{aligned}
&f_1(t) = m-1-t,\ 0 \le t \le m-1,\\
&f_2(0) = m-1,\\
&f_2(t) = m-1-t+(-1)^t,\ 1 \le t \le m-2,\\
&f_2(m-1) = \tfrac{1-(-1)^m}{2},\\
&f_3(t) = m-1-t-(-1)^t,\ 0 \le t \le m-1,
\end{aligned}
$$

If $m \ge 4$ and even $n \ge 4$, $f^j = f_1$ for $0 \le j \le n-1$.

If $3 \le$ odd $m \le$ odd n, $f^j = f_2$ for $0 \le j \le m-1$, $f^j = f_1$ for $m \le j \le n-1$.

If $4 \le$ even $m \le$ odd $n-1$, $f^j = f_{3-j \bmod 2}$ for $0 \le j \le m-1$, $f^j = f_1$ for $m \le j \le n-1$.

For integer l, let $inv(f_l)$ be the inversion number in f_l. By counting the number of crossings in $D_{m,n}$, we have

Lemma 3.1. *If $f^j = f_l$, then $\nu_{D_{m,n}}(E^j) = \binom{m}{2}\binom{m}{2} - (\sum_{t=0}^{m-1}((m-1-t)f_l(t) + t(m-1-f_l(t)) - inv(f_l))$.*

By Lemma 3.1, we can get Lemmas 3.2-3.5:

Lemma 3.2. *If $f^j = f_1$, then $\nu_{D_{m,n}}(E^j) = \frac{m(m-1)(m-2)(3m-5)}{12}$.*

Proof.

$$\begin{aligned}\nu_{D_{m,n}}(E^j) &= \tbinom{m}{2}\tbinom{m}{2} - (\textstyle\sum_{t=0}^{m-1}((m-1-t)(m-1-t) - t(m-1-(m-1-t))) - \tbinom{m}{2}) \\ &= \tfrac{m(m-1)(m-2)(3m-5)}{12}.\end{aligned}$$

□

Lemma 3.3. *If $f^j = f_2$ and m is odd, then $\nu_{D_{m,n}}(E^j) = \frac{m(m-1)(m-2)(3m-5)}{12} + \frac{m-1}{2}$.*

Proof.

$$\begin{aligned}\nu_{D_{m,n}}(E^j) &= \tbinom{m}{2}\tbinom{m}{2} - ((m-1)^2 + 2\textstyle\sum_{t=1}^{\frac{m-1}{2}}((m-1-2t)(m-2t) + 2t(2t-1)) \\ &\quad -(m-1+2\textstyle\sum_{t=1}^{\frac{m-3}{2}} 2t)) \\ &= \tfrac{m(m-1)(m-2)(3m-5)}{12} + \tfrac{m-1}{2}.\end{aligned}$$

□

Lemma 3.4. *If $f^j = f_2$ and m is even, then $\nu_{D_{m,n}}(E^j) = \frac{m(m-1)(m-2)(3m-5)}{12} + \frac{m-2}{2}$.*

Proof.

$$\begin{aligned}\nu_{D_{m,n}}(E^j) &= \tbinom{m}{2}\tbinom{m}{2} - (2(m-1)^2 + 2\textstyle\sum_{t=1}^{\frac{m-2}{2}}((m-1-2t)(m-2t) + 2t(2t-1)) \\ &\quad -(m-1+2\textstyle\sum_{t=1}^{\frac{m-2}{2}}(2t-1))) \\ &= \tfrac{m(m-1)(m-2)(3m-5)}{12} + \tfrac{m-2}{2}.\end{aligned}$$

□

Lemma 3.5. *If $f^j = f_3$ and m is even, then $\nu_{D_{m,n}}(E^j) = \frac{m(m-1)(m-2)(3m-5)}{12} + \frac{m}{2}$.*

Proof.

$$\begin{aligned}\nu_{D_{m,n}}(E^j) &= \tbinom{m}{2}\tbinom{m}{2} - (2\textstyle\sum_{t=1}^{\frac{m}{2}}((m-2t)(m-(2t-1)) + (2t-1)(2t-2)) - 2\textstyle\sum_{t=1}^{\frac{m-2}{2}}(2t)) \\ &= \tfrac{m(m-1)(m-2)(3m-5)}{12} + \tfrac{m}{2}.\end{aligned}$$

□

By Lemma 1.1 and Lemmas 3.2-3.5, we have

Lemma 3.6. *For $m \geq 4$ and even $n \geq 4$,* $\nu(D_{m,n}) = \frac{n \cdot m(m-1)(m-2)(3m-5)}{12}$.

Lemma 3.7. *For $4 \leq m \leq$ odd n,* $\nu(D_{m,n}) = \frac{n \cdot m(m-1)(m-2)(3m-5)}{12} + \frac{m(m-1)}{2}$.

Now we consider the case of $m >$ odd $n \geq 3$. Let $r = \lfloor \frac{m}{n} \rfloor$, $s = m \bmod n$, $s_0 = \frac{n-1}{2}$ and $s_1 = \frac{s}{2}$. Let

$$
\begin{aligned}
&f_4(d \cdot s_0) = m - d \cdot s_0 - 2 + \tfrac{1-(-1)^{s_0}}{2},\ 0 \leq d \leq r-1,\\
&f_4(t + d \cdot s_0) = m - d \cdot s_0 - t - 1 - (-1)^{t+s_0},\ n \geq 5,\ 0 \leq t \leq s_0 - 1,\ 0 \leq d \leq r-1,\\
&f_4(s_0 \cdot r) = s_0 \cdot r + s + r - 2 + \tfrac{1-(-1)^{s_1}}{2},\ s \geq 2,\\
&f_4(t) = m - t - 1 - (-1)^{t+s_1+s_0 * r},\ s \geq 4,\ s_0 \cdot r + 1 \leq t \leq s_0 \cdot r + s_1 - 1,\\
&f_4(s_0 \cdot r + s_1 + d) = (d+1)s_0 - 1,\ 0 \leq d \leq r-1,\\
&f_4(t) = m - t - 1 - (-1)^{t+s_1+r+s_0 * r},\ \text{ even } s \geq 4,\ s_0 \cdot r + s_1 + r \leq t \leq s_0 \cdot r + s + r - 2,\\
&f_4(s_0 \cdot r + s + r - 1) = s_0 \cdot r + 1 - \tfrac{1-(-1)^{s_1}}{2},\ \text{ even } s \geq 2,\\
&f_4(s_0 \cdot r + r) = s_0 \cdot r + r,\ s = 1,\\
&f_4(s_0 \cdot r + s_1 + r) = s_0 \cdot r + s_1 - 1,\ \text{ odd } s \geq 3,\\
&f_4(s_0 \cdot r + s_1 + r + 1) = s_0 \cdot r + s_1 + r,\ \text{ odd } s \geq 3,\\
&f_4(t) = m - t - 1 - (-1)^{t+s_1+r+s_0 * r},\ \text{ odd } s \geq 7,\ s_0 \cdot r + s_1 + r + 2 \leq t \leq s_0 \cdot r + s + r - 2,\\
&f_4(s_0 \cdot r + s + r - 1) = s_0 \cdot r + s_1 + \tfrac{1-(-1)^{s_1}}{2},\ \text{ odd } s \geq 5,\\
&f_4(m - (d+1)s_0) = s_0 \cdot r + s_1 + d,\ 0 \leq d \leq r-1,\\
&f_4(t) = m - t - 1 + (-1)^{t+r+s_0+s_0 \cdot d},\ n \geq 5,\ m - d \cdot s_0 - s_0 + 1 \leq t \leq m - d \cdot s_0 - 2,\ 0 \leq d \leq r-1,\\
&f_4(m - d \cdot s_0 - 1) = d \cdot s_0 + \tfrac{1-(-1)^{s_0}}{2},\ n \geq 5,\ 0 \leq d \leq r-1.
\end{aligned}
$$

$$
\begin{aligned}
&f_5(t) = f_4(t),\ 0 \leq t \leq s_0 \cdot r - 1 \text{ or } m - s_0 \cdot r \leq t \leq m-1,\\
&f_5(s_0 \cdot r) = s_0 \cdot r + s + r - 1 - \tfrac{1-(-1)^{s_1}}{2},\ s \geq 4,\\
&f_5(t) = m - t - 1 + (-1)^{s_0 \cdot r + s_1 + t},\ s \geq 6,\ s_0 \cdot r + 1 \leq t \leq s_0 \cdot r + s_1 - 2,\\
&f_5(s_0 \cdot r + s_1 - 1) = s_0 \cdot r + s_1 - 1,\ s \geq 2,\\
&f_5(s_0 \cdot r + s_1 + d) = (d+1)s_0 - 1,\ 0 \leq d \leq r-1,\\
&f_5(s_0 \cdot r + s_1 + r) = s_0 \cdot r + s_1 + r,\ s \geq 2\\
&f_5(t) = m - t - 1 - (-1)^{t+s_1+r+s_0 \cdot r},\ s \geq 6,\ s_0 \cdot r + s_1 + r + 1 \leq t \leq s_0 \cdot r + s + r - 2,\\
&f_5(s_0 \cdot r + s + r - 1) = s_0 \cdot r + \tfrac{1-(-1)^{s_1}}{2},\ s \geq 4.
\end{aligned}
$$

$$
\begin{aligned}
&f_6(t) = f_4(t),\ 0 \leq t \leq s_0 \cdot r - 1 \text{ or } m - s_0 \cdot r \leq t \leq m-1,\\
&f_6(s_0 \cdot r) = s_0 \cdot r + s + r - 1,\ s \geq 1,\\
&f_6(t) = m - t - 1,\ s \geq 3,\ s_0 \cdot r + 1 \leq t \leq s_0 \cdot r + s_1 - 1 + \tfrac{1-(-1)^{s}}{2}\\
&f_6(s_0 \cdot r + s_1 + d + \tfrac{1-(-1)^{s}}{2}) = (d+1)s_0 - 1,\ 0 \leq d \leq r-1,\\
&f_6(t) = m - t - 1,\ s \geq 2,\ s_0 \cdot r + s_1 + r + \tfrac{1-(-1)^{s}}{2} \leq t \leq s_0 \cdot r + s + r - 1.
\end{aligned}
$$

$$
\begin{aligned}
&f_7(t) = f_4(t),\ 0 \leq t \leq s_0 \cdot r - 1 \text{ or } m - s_0 \cdot r \leq t \leq m-1,\\
&f_7(s_0 \cdot r) = s_0 \cdot r + s + r - 1,\ s \geq 3,\\
&f_7(t) = m - t - 1,\ s \geq 5,\ s_0 \cdot r + 1 \leq t \leq s_0 \cdot r + s_1 - 1,\\
&f_7(s_0 \cdot r + d + s_1) = (d+1)s_0 - 1,\ 0 \leq d \leq r-1,\\
&f_7(t) = m - t - 1,\ s_0 \cdot r + s_1 + r \leq t \leq s_0 \cdot r + s + r - 1,\\
&f_7(m - (d+1)s_0) = s_0 \cdot r + s_1 + d + 1,\ 0 \leq d \leq r-1,\\
&f_7(t) = m - t - 1 + (-1)^{t+r+s_0+s_0 \cdot d},\ m - d \cdot s_0 - s_0 + 1 \leq t \leq m - d \cdot s_0 - 2,\ 0 \leq d \leq r-1,\\
&f_7(m - d \cdot s_0 - 1) = d \cdot s_0 + \tfrac{1-(-1)^{s_0}}{2},\ 0 \leq d \leq r-1.
\end{aligned}
$$

$$
\begin{aligned}
&f_8(t) = f_4(t),\ 0 \leq t \leq s_0 \cdot r - 1,\\
&f_8(s_0 \cdot r) = s_0 \cdot r + s + r - 1, s \geq 1,\\
&f_8(t) = m - t - 1,\ s \geq 3,\ s_0 \cdot r + 1 \leq t \leq s_0 \cdot r + s_1 - 1,\\
&f_8(s_0 \cdot r + d + s_1) = (d+1)s_0 - 1,\ 0 \leq d \leq r-1,\\
&f_8(t) = m - t - 1,\ s \geq 2,\ s_0 \cdot r + s_1 + r \leq t \leq s_0 \cdot r + s + r - 1,\\
&f_8(m - (d+1)s_0) = s_0 \cdot r + s_1 + d + 1,\ 0 \leq d \leq r-1,\\
&f_8(t) = m - t - 1 + (-1)^{t+r+s_0+s_0 \cdot d},\ m - d \cdot s_0 - s_0 + 1 \leq t \leq m - d \cdot s_0 - 2,\ 0 \leq d \leq r-1,\\
&f_8(m - d \cdot s_0 - 1) = d \cdot s_0 + \tfrac{1-(-1)^{s_0}}{2},\ 0 \leq d \leq r-1.
\end{aligned}
$$

If s is even, $f^j = f_{5-j \bmod 2}$ for $0 \leq j \leq s-1$, $f^j = f_6$ for $s \leq j \leq n-1$.

If s is odd, $f^j = f_4$ for $0 \leq j \leq s-1$, $f^j = f_{8-j \bmod 2}$ for $s \leq j \leq n-1$.

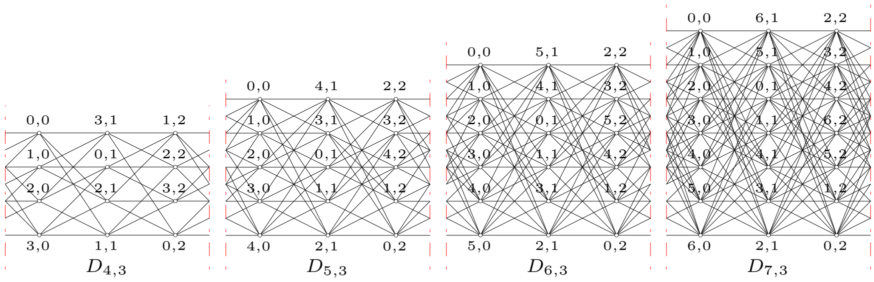

Figure 3.3: Drawings $D_{m,3}$ for $4 \leq m \leq 7$

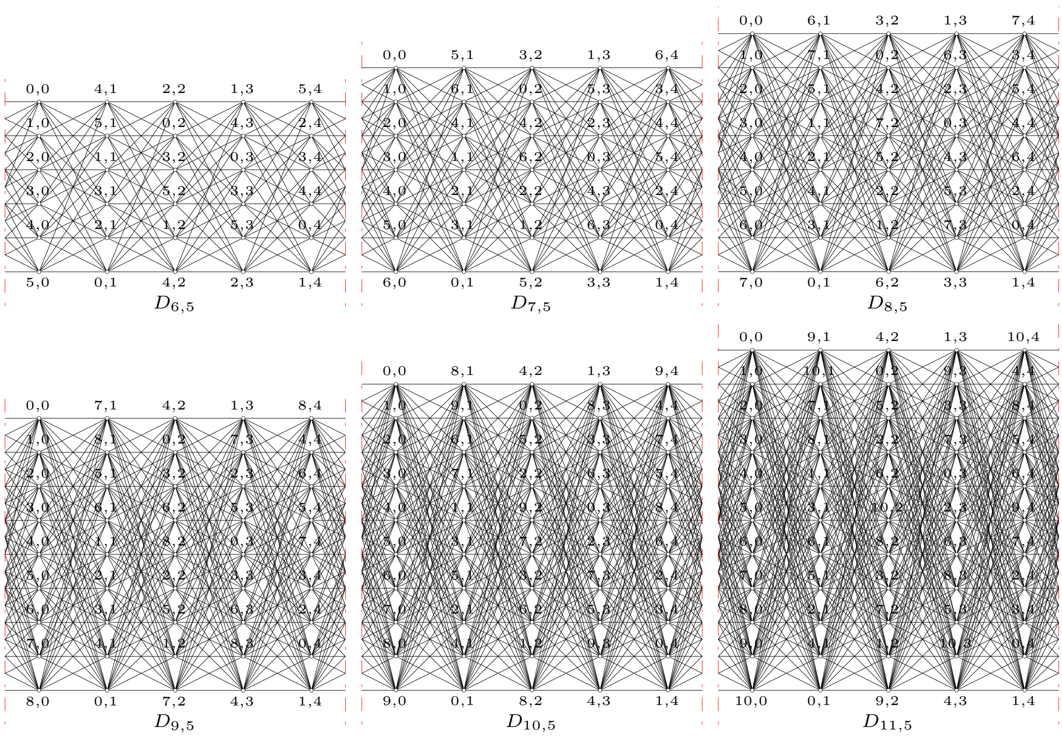

Figure 3.4: Drawings $D_{m,5}$ for $6 \leq m \leq 11$

For $0 \leq i < j \leq m-1$, let

$$inv_{l,i,j} \quad = \begin{cases} 1 & \text{if } f_l(i) > f_l(j) \\ 0 & \text{if } f_l(i) < f_l(j). \end{cases}$$

Then $inv(f_l) = \sum_{i=0}^{m-1} \sum_{j=i+1}^{m-1} (inv_{l,i,j})$.

For $l = 4, 5, 6, 7, 8,$ let

$$S_1 = \quad \{i \mid 0 \leq i \leq s_0 \cdot r - 1\} \bigcup_{d=1}^{r} \{i \mid m + 1 - d \cdot s_0 \leq m - 1 - (d-1)d_0\}.$$

For $l = 4, 5, 6, 7,$ let

$$S_2 = \quad \{i \mid s_0 \cdot r + s_1 \leq i \leq s_0 \cdot r + s_1 + r - 1\} \bigcup_{d=1}^{r} \{m - 1 - d \cdot s_0 + 1\},$$

$$S_3 = \quad \{i \mid s_0 \cdot r \leq i \leq s_0 \cdot r + s_1 - 1\} \bigcup \{i \mid s_0 \cdot r + s_1 + r \leq i \leq s_0 \cdot r + s\}.$$

for $l = 8$, let

$$\begin{aligned} S_2 = &\ \{i \mid s_0 \cdot r + s_1 + 1 \leq i \leq s_0 \cdot r + s_1 + r\} \bigcup_{d=1}^{r} \{m - 1 - d \cdot s_0 + 1\}, \\ S_3 = &\ \{i \mid s_0 \cdot r \leq i \leq s_0 \cdot r + s_1\} \bigcup \{i \mid s_0 \cdot r + s_1 + r + 1 \leq i \leq s_0 \cdot r + s\}. \end{aligned}$$

For $l = 4, 5, 6, 7, 8$ and $k = 1, 2, 3$, let

$$F_{l,k} = \sum_{t \in S_k} (m-1-t) f_l(t) + t(m-1-f_l(t)) - \sum_{t \in S_k} \sum_{j=t+1}^{m-1} (inv_{l,t,j}).$$

By Lemma 3.1, we have

Lemma 3.8. *If $f^j = f_l$, then $\nu_{D_{m,n}}(E^j) = \binom{m}{2}\binom{m}{2} - (F_{l,1} + F_{l,2} + F_{l,3})$.*

By the definition of f_l and $F_{l,k}$, we have Lemmas 3.9-3.11:

Lemma 3.9. *For $l = 4, 5, 6, 7, 8$,*

$$F_{l,1} = \frac{4(2s_0^3 - s_0^2)r^3 - 3((4m+2)s_0^2 - 2m \cdot s_0 + 1)r^2 + (-2s_0^2 + (12m^2 - 12m + 10)s_0 - 6m^2 - 3)r}{6}.$$

Proof. For even s_0, we have

$$\begin{aligned} F_{l,1} = &\ 2\sum_{t=1}^{\frac{s_0 \cdot r}{2}} ((m-2t)(m-2t+1) + (2t-1)(2t-2)) \\ &+ \sum_{d=1}^{r} (2\sum_{t=1}^{\frac{s_0-2}{2}} ((s_0 \cdot d - 2t)(s_0 \cdot d - 2t - 1) + (m - s_0 \cdot d + 2t - 1)(m - s_0 \cdot d + 2t))) \\ &+ \sum_{d=1}^{r} ((m - 1 + s_0 - s_0 \cdot d)^2 + (s_0(d-1))^2) \\ &- (2\sum_{t=1}^{\frac{s_0 \cdot r}{2}} (m - 2t) + \sum_{d=1}^{r} (2\sum_{t=1}^{\frac{s_0-2}{2}} (2t - 1 + (s_0-1)(d-1)) + \sum_{d=1}^{r} (s_0-1)(d-1))) \\ = &\ \frac{4(2s_0^3 - s_0^2)r^3 - 3((4m+2)s_0^2 - 2(m+2)s_0 + 1)r^2 + (-2s_0^2 + (12m^2 - 12m + 10)s_0 - 6m^2 - 3)r}{6}. \end{aligned}$$

For odd s_0, we have

$$\begin{aligned} F_{l,1} = &\ \sum_{d=1}^{r} ((m - 1 + s_0 - s_0 \cdot d)^2 + (s_0(d-1))^2) \\ &+ \sum_{d=1}^{r} (2\sum_{t=1}^{\frac{s_0-1}{2}} ((m - s_0 \cdot d + 2t - 1)(m - s_0 \cdot d + 2t - 2) + (s_0 \cdot d - 2t)(s_0 \cdot d - 2t + 1))) \\ &+ \sum_{d=1}^{r} (2\sum_{t=1}^{\frac{s_0-1}{2}} ((s_0 \cdot d + 2t - 1)(s_0 \cdot d + 2t - 2) + (m - s_0 \cdot d - 2t)(m - s_0 \cdot d - 2t + 1))) \\ &- (\sum_{d=1}^{r} (m - 1 + s_0 - s_0 \cdot d) + \sum_{d=1}^{r} (2\sum_{t=1}^{\frac{s_0-1}{2}} (m - 1 + s_0 - s_0 \cdot d - 2t)) + \sum_{d=1}^{r} (2\sum_{t=1}^{\frac{s_0-1}{2}} ((s_0-1)(d-1) - 2 + 2t))) \\ = &\ \frac{4(2s_0^3 - s_0^2)r^3 - 3((4m+2)s_0^2 - 2(m+2)s_0 + 1)r^2 + (-2s_0^2 + (12m^2 - 12m + 10)s_0 - 6m^2 - 3)r}{6}. \end{aligned}$$

□

Lemma 3.10.

$$F_{l,2} = \begin{cases} \frac{-(2s_0-1)r^2 + (-2s_0 + 2m^2 - 4m + 5)r}{2} & \text{if } l = 4, 5, 6 \\ \frac{r^2 + (2m^2 - 6m + 3)r}{2} & \text{if } l = 7, \\ \frac{(-4s_0+1)r^2 + (-4s_0 + 2m^2 - 2m + 7)r}{2} & \text{if } l = 8. \end{cases}$$

Proof. For $l = 4, 5, 6$, we have

$$\begin{aligned} F_{l,2} = &\ \sum_{d=1}^{r} ((m - s_0 \cdot r - s_1 - d)(s_0 \cdot d - 1) + (s_0 \cdot r + s_1 + d - 1)(m - s_0 \cdot d)) \\ &+ \sum_{d=1}^{r} ((s_0 \cdot d - 1)(s_0 \cdot r + s_1 + d - 1) + (m - s_0 \cdot d)(m - s_0 \cdot r - s_1 - d)) \\ &- (\sum_{d=1}^{r} d(s_0 - 1) + \sum_{d=1}^{r} (s_0 \cdot d - 1)) \\ = &\ \frac{-(2s_0-1)r^2 + (-2s_0 + 2m^2 - 4m + 5)r}{2}. \end{aligned}$$

For $l = 7$, we have

$$\begin{aligned} F_{7,2} = \ & \textstyle\sum_{d=1}^{r}((m - s_0 \cdot r - s_1 - d)(s_0 \cdot d - 1) + (s_0 \cdot r + s_1 + d - 1)(m - s_0 \cdot d)) \\ & + \textstyle\sum_{d=1}^{r}((s_0 \cdot d - 1)(s_0 \cdot r + s_1 + d) + (m - s_0 \cdot d)(m - s_0 \cdot r - s_1 - d - 1)) \\ & -(\textstyle\sum_{d=1}^{r} d(s_0 - 1) + \sum_{d=1}^{r}(s_0 \cdot d - 1)) \\ = \ & \frac{r^2 + (2m^2 - 6m + 3)r}{2}. \end{aligned}$$

For $l = 8$, we have

$$\begin{aligned} F_{8,2} = \ & \textstyle\sum_{d=1}^{r}((m - 1 - s_0 \cdot r - s_1 - d)(s_0 \cdot d - 1) + (s_0 \cdot r + s_1 + d)(m - s_0 \cdot d)) \\ & + \textstyle\sum_{d=1}^{r}((s_0 \cdot d - 1)(s_0 \cdot r + s_1 + d - 1) + (m - s_0 \cdot d)(m - s_0 \cdot r - s_1 - d)) \\ & -(\textstyle\sum_{d=1}^{r}(s_0 - 1)d + \sum_{d=1}^{r}(s_0 \cdot d - 1)) \\ = \ & \frac{(-4s_0 + 1)r^2 + (-4s_0 + 2m^2 - 2m + 7)r}{2}. \end{aligned}$$

□

Lemma 3.11.

$$F_{l,3} = \begin{cases} \frac{12s_0^2 s_1 \cdot r^2 + 3(4s_0 \cdot s_1^2 - (4m \cdot s_0 - 1)s_1)r + 4s_1^3 - 6m \cdot s_1^2 + (6m^2 - 9m + 2)s_1 + 3\frac{1-(-1)^{s_1}}{2}}{3} & \text{if } l = 4 \text{ and } s \geq 2 \text{ is even}, \\ \frac{12(s_0^2 s_1 - 2s_0^2)r^2 + 3(4s_0 s_1^2 - (4(m+4)s_0 - 1)s_1 + (8m+10)s_0 + 1)r}{3} \\ \quad + \frac{4s_1^3 - 6(m+4)s_1^2 + (6m^2 + 15m + 32)s_1 - (6m^2 + 15m) + 3\frac{1+(-1)^{s_1}}{2}}{3} & \text{if } l = 5 \text{ and } s \geq 2 \text{ is even}, \\ \frac{12s_0^2 s_1 \cdot r^2 + 3(4s_0 \cdot s_1^2 - (4m \cdot s_0 - 1)s_1)r + 4s_1^3 - 6m \cdot s_1^2 + (6m^2 - 9m + 5)s_1}{3} & \text{if } l = 6 \text{ and } s \text{ is even}, \\ -2(s_0^2 + 2s_0 + 1)r^2 + ((2m - 3)s_0 + 2(m - 1))r & \text{if } l = 4 \text{ and } s = 1, \\ \frac{6(2s_0^2 \cdot s_1 + s_0^2 + s_0)r^2 + 3(4s_0 \cdot s_1^2 - (4(m-1)s_0 - 3)s_1 - (2m-1)s_0 - m - 1)r}{3} \\ \quad + \frac{4s_1^3 - 6(m-1)s_1^2 + (6m^2 - 15m + 5)s_1 + 3m^2 - 6m + 3}{3} & \text{if } l = 4 \text{ and } s \geq 3 \text{ is odd}, \\ \frac{6(2s_0^2 \cdot s_1 + s_0^2)r^2 + 6(2s_0 \cdot s_1^2 - 2(m-1)s_0 \cdot s_1 - (m-1)s_0)r + 4s_1^3 - 6(m-1)s_1^2 + (6m^2 - 12m + 8)s_1 + 3m^2 - 6m + 3}{3} & \text{if } l = 7 \text{ and } s \text{ is odd}, \\ \frac{6(2s_0^2 \cdot s_1 + s_0^2)r^2 + 3(4s_0 \cdot s_1^2 - (4(m-1)s_0 - 1)s_1 - (2m-3)s_0)r + 4s_1^3 - 6(m-1)s_1^2 + (6m^2 - 15m + 14)s_1 + 3m^2 - 9m + 6}{3} & \text{if } l = 8 \text{ and } s \text{ is odd}. \end{cases}$$

Proof. We first consider the cases of even s. For $l = 4$ and even $s_1 \geq 2$, we have

$$\begin{aligned} F_{4,3} = \ & 4\textstyle\sum_{t=1}^{\frac{s_1}{2}}((m - s_0 \cdot r - 2t)(m - s_0 \cdot r - 2t + 1) + (s_0 \cdot r + 2t - 1)(s_0 \cdot r + 2t - 2)) \\ & -(2\textstyle\sum_{t=1}^{\frac{s_1}{2}}(m - s_0 \cdot r - 2t) + 2\sum_{t=1}^{\frac{s_1}{2}}(s_0 - 1)r - 2 + 2t)) \\ = \ & \frac{12s_0^2 s_1 \cdot r^2 + 3(4s_0 \cdot s_1^2 - (4m \cdot s_0 - 1)s_1)r + 4s_1^3 - 6m \cdot s_1^2 + (6m^2 - 9m + 2)s_1}{3}. \end{aligned}$$

For $l = 4$ and odd s_1, we have

$$\begin{aligned} F_{4,3} = \ & 2((m - 1 - s_0 \cdot r)^2 + (s_0 \cdot r)^2) \\ & +4\textstyle\sum_{t=1}^{\frac{s_1-1}{2}}((m - 1 - s_0 \cdot r - 2t)(m - s_0 \cdot r - 2t) + (s_0 \cdot r + 2t)(s_0 \cdot r - 1 + 2t)) \\ & -(m - 1 - s_0 \cdot r + 2\textstyle\sum_{t=1}^{\frac{s_1-1}{2}}(m - 1 - s_0 \cdot r - 2t) + 2\sum_{t=1}^{\frac{s_1-1}{2}}((s_0 - 1)r - 1 + 2t) + s_0 r - r) \\ = \ & \frac{12s_0^2 s_1 \cdot r^2 + 3(4s_0 \cdot s_1^2 - (4m \cdot s_0 - 1)s_1)r + 4s_1^3 - 6m \cdot s_1^2 + (6m^2 - 9m + 2)s_1 + 3}{3}. \end{aligned}$$

For $l = 5$ and even $s_1 \geq 2$, we have

$$\begin{aligned} F_{5,3} = \ & 2((m - 1 - s_0 \cdot r)^2 + (s_0 \cdot r)^2) \\ & +4\textstyle\sum_{t=1}^{\frac{s_1-2}{2}}((m - 1 - s_0 \cdot r - 2t)(m - s_0 \cdot r - 2t) + (s_0 \cdot r + 2t)(s_0 \cdot r - 1 + 2t)) \\ & +2((m - s_0 \cdot r - s_1)(s_0 \cdot r + s_1 - 1) + (s_0 \cdot r + s_1 - 1)(m - s_0 \cdot r - s1)) \\ & -(m - 1 - s_0 r + 2\textstyle\sum_{t=1}^{\frac{s_1-2}{2}}(m - 1 - s_0 \cdot r - 2t) + 2(m - 1 - s_0 \cdot r - s_1 - r) + 2\sum_{t=1}^{\frac{s_1-2}{2}}(s_0 - 1)r - 1 + 2t) + (s0 - 1)r) \\ = \ & \frac{12(s_0^2 s_1 - 2s_0^2)r^2 + 3(4s_0 s_1^2 - (4(m+4)s_0 - 1)s_1 + (8m+10)s_0 + 1)r + 4s_1^3 - 6(m+4)s_1^2 + (6m^2 + 15m + 32)s_1 - (6m^2 + 15m) + 3}{3}. \end{aligned}$$

For $l = 5$ and odd s_1, we have

$$\begin{aligned} F_{5,3} = \ & 4\textstyle\sum_{t=1}^{\frac{s_1-1}{2}}((m - s_0 \cdot r - 2t)(m - s_0 \cdot r - 2t + 1) + (s_0 \cdot r + 2t - 1)(s_0 \cdot r + 2t - 2)) \\ & +2((m - s_0 \cdot r - s_1)(s_0 \cdot r + s_1 - 1) + (s_0 \cdot r + s_1 - 1)(m - s_0 \cdot r - s1)) \\ & -(2\textstyle\sum_{t=1}^{\frac{s_1-1}{2}}(m - s_0 r - 2t) + 2(m - 1 - s_0 \cdot r - s_1 - r) + 2\sum_{t=1}^{\frac{s_1-1}{2}}((s_0 - 1)r - 2 + 2t)) \\ = \ & \frac{12(s_0^2 s_1 - 2s_0^2)r^2 + 3(4s_0 s_1^2 - (4(m+4)s_0 - 1)s_1 + (8m+10)s_0 + 1)r + 4s_1^3 - 6(m+4)s_1^2 + (6m^2 + 15m + 32)s_1 - (6m^2 + 15m)}{3}. \end{aligned}$$

For $l = 6$, we have

$$\begin{aligned} F_{6,3} = \ & 2\textstyle\sum_{t=1}^{s_1}((m - s_0 \cdot r - t)(m - s_0 \cdot r - t) + (s_0 \cdot r + t - 1) + (s_0 \cdot r + t - 1) \\ & -\textstyle\sum_{t=1}^{s_1}(m - s_0 r - t) + \sum_{t=1}^{s_1}(s_0 - 1)r + t - 1) \\ = \ & \frac{12s_0^2 s_1 \cdot r^2 + 3(4s_0 \cdot s_1^2 - (4m \cdot s_0 - 1)s_1)r + 4s_1^3 - 6m \cdot s_1^2 + (6m^2 - 9m + 5)s_1}{3}. \end{aligned}$$

Now, we consider the cases of odd s. For $l = 4$ and even $s_1 = 0$ ($s = 1$), we have

$$\begin{aligned}
F_{4,3} = \quad & (m-1-s_0\cdot r-r)(s_0\cdot r+r)+(s0\cdot r+r)(m-1-s_0\cdot r-r)-s_0\cdot r \\
= \quad & -2(s_0^2+2s_0+1)r^2+((2m-3)s_0+2(m-1))r.
\end{aligned}$$

For $l = 4$ and even $s_1 \geq 2$, we have

$$\begin{aligned}
F_{4,3} = \quad & 2\sum_{t=1}^{\frac{s_1}{2}}((m-s_0\cdot r-2t)(m-s_0\cdot r-2t+1)+(s_0\cdot r+2t-1)(s_0\cdot r+2t-2)) \\
& +(s_0\cdot r+s_1)(s_0\cdot r+s_1-1)+(m-1-s_0\cdot r-s_1)(m-s_0\cdot r-s_1) \\
& +(s_0\cdot r+s_1-1)(s_0\cdot r+s_1+r)+(m-s_0\cdot r-s_1)(m-1-s_0\cdot r-s_1-r) \\
& +2\sum_{t=1}^{\frac{s_1-2}{2}}((s_0\cdot r+s_1-2t)(s_0\cdot r+s_1-2t-1)+(m-1-s_0\cdot r-s_1+2t)(m-s_0\cdot r-s_1+2t)) \\
& +(s_0\cdot r)^2+(m-1-s_0\cdot r)^2 \\
& -(2\sum_{t=1}^{\frac{s_1}{2}}(m-s_0\cdot r-2t)+(s_0-1)r+s_1-1+s_0\cdot r+s_1-1+2\sum_{t=1}^{\frac{s_1-2}{2}}((s_0-1)r+2t-1)+(s_0-1)r) \\
= \quad & \frac{6(2s_0^2\cdot s_1+s_0^2+s_0)r^2+3(4s_0\cdot s_1^2-(4(m-1)s_0-3)s_1-(2m-1)s_0-m-1)r+4s_1^3-6(m-1)s_1^2+(6m^2-15m+5)s_1+3m^2-6m+3}{3}.
\end{aligned}$$

For $l = 4$ and odd s_1, we have

$$\begin{aligned}
F_{4,3} = \quad & (m-1-s_0\cdot r)^2+(s_0\cdot r)^2 \\
& +2\sum_{t=1}^{\frac{s_1-1}{2}}((m-1-s_0\cdot r-2t)(m-s_0\cdot r-2t)+(s_0\cdot r+2t)(s_0\cdot r-1+2t)) \\
& +(s_0\cdot r+s_1)(s_0\cdot r+s_1-1)+(m-1-s_0\cdot r-s_1)(m-s_0\cdot r-s_1) \\
& +(s_0\cdot r+s_1-1)(s_0\cdot r+s_1+r)+(m-s_0\cdot r-s_1)(m-1-s_0\cdot r-s_1-r) \\
& +2\sum_{t=1}^{\frac{s_1-1}{2}}((s_0\cdot r+s_1-2t)(s_0\cdot r+s_1-2t-1)+(m-1-s_0\cdot r-s_1+2t)(m-s_0\cdot r-s_1+2t)) \\
& -(m-1-s_0\cdot r+2\sum_{t=1}^{\frac{s_1-1}{2}}(m-s_0\cdot r-2t)+(s_0-1)r+s_1-1+s_0\cdot r+s_1-1+2\sum_{t=1}^{\frac{s_1-1}{2}}((s_0-1)r+2t-2)) \\
= \quad & \frac{6(2s_0^2\cdot s_1+s_0^2+s_0)r^2+3(4s_0\cdot s_1^2-(4(m-1)s_0-3)s_1-(2m-1)s_0-m-1)r+4s_1^3-6(m-1)s_1^2+(6m^2-15m+5)s_1+3m^2-6m+3}{3}.
\end{aligned}$$

For $l = 7$, we have

$$\begin{aligned}
F_{7,3} = \quad & \sum_{t=1}^{s_1}((m-s_0\cdot r-t)(m-s_0\cdot r-t)+(s_0\cdot r+t-1)+(s_0\cdot r+t-1) \\
& +\sum_{t=1}^{s_1+1}((m-s_0\cdot r-t)(m-s_0\cdot r-t)+(s_0\cdot r+t-1)+(s_0\cdot r+t-1) \\
& -\sum_{t=1}^{s1}(m-s_0r-t)+\sum_{t=1}^{s_1+1}(s_0-1)r+t-1) \\
= \quad & \frac{6(2s_0^2\cdot s_1+s_0^2)r^2+3(4s_0\cdot s_1^2-(4(m-1)s_0-1)s_1-(2m-1)s_0+1)r+4s_1^3-6(m-1)s_1^2+(6m^2-15m+8)s_1+3m^2-6m+3}{3}.
\end{aligned}$$

For $l = 8$, we have

$$\begin{aligned}
F_{8,3} = \quad & \sum_{t=1}^{s_1+1}((m-s_0\cdot r-t)(m-s_0\cdot r-t)+(s_0\cdot r+t-1)+(s_0\cdot r+t-1) \\
& +\sum_{t=1}^{s_1}((m-s_0\cdot r-t)(m-s_0\cdot r-t)+(s_0\cdot r+t-1)+(s_0\cdot r+t-1) \\
& -\sum_{t=1}^{s1+1}(m-s_0r-t)+\sum_{t=1}^{s_1}(s_0-1)r+t-1) \\
= \quad & \frac{6(2s_0^2\cdot s_1+s_0^2)r^2+3(4s_0\cdot s_1^2-(4(m-1)s_0-1)s_1-(2m-3)s_0)r+4s_1^3-6(m-1)s_1^2+(6m^2-15m+14)s_1+3m^2-9m+6}{3}.
\end{aligned}$$

□

By Lemmas 3.8-3.11, we have

Lemma 3.12. *For $m >$ odd $n \geq 3$,*

$$\nu(D_{m,n}) \quad = \begin{cases} \frac{m(m-1)(m-2)(3m-5)n+2m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-n^2-7m-n+5+(2m\cdot n+4m+13n+8)(\frac{m-1}{n})^2}{12} & \text{if } s=1 \\ \frac{m(m-1)(m-2)(3m-5)n+(2n^3-4n^2-8n)\lfloor\frac{m}{n}\rfloor^3-(6m\cdot n^2-3n^3-6m\cdot n-12m+15n)\lfloor\frac{m}{n}\rfloor^2}{12} & \\ +\frac{(6m^2n-6m\cdot n^2+n^3-6m\cdot n+4n^2+24m-13n+3n(1-(-1)^{m-n\lfloor\frac{m}{n}\rfloor}))\lfloor\frac{m}{n}\rfloor+6m^2-6m}{12} & \text{if } s\neq 1. \end{cases}$$

Proof. For even s, we have

$$\begin{aligned}
& \nu(D_{m,n}) \\
= \quad & s_1(\tbinom{m}{2}\tbinom{m}{2}-(F_{4,1}+F_{4,2}+F_{4,3})+\tbinom{m}{2}\tbinom{m}{2}-(F_{5,1}+F_{5,2}+F_{5,3}))+(n-2s_1)(\tbinom{m}{2}\tbinom{m}{2}-(F_{6,1}+F_{6,2}+F_{6,3})) \\
= \quad & \frac{(3n\cdot m^2(m-1)^2-8n(2s_0^3-s_0^2)r^3-6(8n\cdot s_0^2\cdot s_1-(4m\cdot n-2n)s_0^2+2(m\cdot n+n+4s_1+2)s_0)r^2}{12} \\
& -\frac{2(24n\cdot s_0\cdot s_1^2-6(4(m\cdot n+4s_1)s_0-n)s_1-2n\cdot s_0^2+(12n\cdot m^2-12(n-4s_1)m+4n+60s_1)s_0-12n\cdot m+12n+6s_1)r}{12} \\
& -\frac{2(8n\cdot s_1^3-12(m\cdot n+4s_1)s_1^2+(12n\cdot m^2-(18n-48s_1)m+10n+48s_1)s_1-(12m^2+30m-6)s_1)}{12} \\
= \quad & \frac{m(m-1)(m-2)(3m-5)n+(2n^3-4n^2-8n)\lfloor\frac{m}{n}\rfloor^3-(6m\cdot n^2-3n^3-6m\cdot n-12m+15n)\lfloor\frac{m}{n}\rfloor^2}{12} \\
& +\frac{(6m^2n-6m\cdot n^2+n^3-6m\cdot n+4n^2+24m-13n+3n(1-(-1)^{m-n\lfloor\frac{m}{n}\rfloor}))\lfloor\frac{m}{n}\rfloor+6m^2-6m}{12}.
\end{aligned}$$

For $s = 1$, we have

$$\begin{aligned}
&\nu(D_{m,n})\\
= \quad & (2s_1+1)(\tbinom{m}{2}\tbinom{m}{2} - (F_{4,1}+F_{4,2}+F_{4,3})) + \tfrac{n-2s_1-1}{2}(\tbinom{m}{2}\tbinom{m}{2} - (F_{7,1}+F_{7,2}+F_{7,3}) + \tbinom{m}{2}\tbinom{m}{2} - (F_{8,1}+F_{8,2}+F_{8,3}))\\
= \quad & \tfrac{3n\cdot m^2(m-1)^2-8n(2s_0^3-s_0^2)r^3-6(-(4m\cdot n-2n+8)s_0^2+2(m\cdot n+n-4)s_0-4)r^2}{12}\\
& -\tfrac{2(-2n\cdot s_0^2+(12m^2\cdot n-24m(n-1)+16n-30)s_0-12m(n-1)+15(n-1))r+12m^2(n-1)-30m(n-1)+18(n-1)}{12}\\
= \quad & \tfrac{m(m-1)(m-2)(3m-5)n+2m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-n^2-7m-n+5+(2m\cdot n+4m+13n+8)(\frac{m-1}{n})^2}{12}
\end{aligned}$$

For odd s, we have

$$\begin{aligned}
&\nu(D_{m,n})\\
= \quad & (2s_1+1)(\tbinom{m}{2}\tbinom{m}{2} - (F_{4,1}+F_{4,2}+F_{4,3})) + \tfrac{n-2s_1-1}{2}(\tbinom{m}{2}\tbinom{m}{2} - (F_{7,1}+F_{7,2}+F_{7,3}) + \tbinom{m}{2}\tbinom{m}{2} - (F_{8,1}+F_{8,2}+F_{8,3}))\\
= \quad & \tfrac{(3n\cdot m^2(m-1)^2-8n(2s_0^3-s_0^2)r^3-6(8n\cdot s_0^2\cdot s_1-(4m\cdot n+2n+16s_1)s_0^2+2n(m+1)s_0)r^2}{12}\\
& -\tfrac{2(24n\cdot s_0\cdot s_1^2-6(4n(m-1)s_0-n-4s_1-2)s_1-2n\cdot s_0^2+(12m^2\cdot n-24m\cdot n+16n-12s_1-6)s_0-m(12n+12s_1+6)+15n-18s_1-9)r}{12}\\
& -\tfrac{2(8n\cdot s_1^3-12n(m-1)s_1^2+(12m^2\cdot n-30m\cdot n+22n-24s_1-12)s_1+6m^2\cdot n-m(15n-6s_1-3)+9n-6s_1-3)}{12}\\
= \quad & \tfrac{m(m-1)(m-2)(3m-5)n+(2n^3-4n^2-8n)\lfloor\frac{m}{n}\rfloor^3-(6m\cdot n^2-3n^3-6m\cdot n-12m+15n)\lfloor\frac{m}{n}\rfloor^2}{12}\\
& +\tfrac{(6m^2n-6m\cdot n^2+n^3-6m\cdot n+4n^2+24m-13n+3n(1-(-1)^{m-n\lfloor\frac{m}{n}\rfloor}))\lfloor\frac{m}{n}\rfloor+6m^2-6m}{12}.
\end{aligned}$$

□

Lemma 3.13. *For $m > n = 3$, $\nu(D_{m,n}) \leq \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{28m^3-54m^2+42m+16}{108}$.*

Proof. By Lemma 3.12, for $s = 1$, we have

$$\begin{aligned}
&\nu(D_{m,n})\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)+2m^3-9m^2+9m+12m-9-7m-3+5+(6m+4m+39+8)(\frac{m-1}{3})^2}{12}\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \tfrac{28m^3-54m^2+42m-16}{108}\\
\leq \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \tfrac{28m^3-54m^2+42m+16}{108}.
\end{aligned}$$

for $s = 0$, we have

$$\begin{aligned}
&\nu(D_{m,n})\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \tfrac{(54-36-24)(\frac{m}{3})^3-(54m-81-18m-12m+45)(\frac{m}{3})^2+(18m^2-54m+27-18m+36+24m-39)\frac{m}{3}+6m^2-6m}{12}\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \tfrac{28m^3-54m^2+18m}{108}\\
\leq \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \tfrac{28m^3-54m^2+42m+16}{108}.
\end{aligned}$$

for $s = 2$, we have

$$\begin{aligned}
&\nu(D_{m,n})\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \tfrac{(54-36-24)(\frac{m-2}{3})^3-(54m-81-18m-12m+45)(\frac{m-2}{3})^2+(18m^2-54m+27-18m+36+24m-39)\frac{m-2}{3}+6m^2-6m}{12}\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \tfrac{28m^3-54m^2+42m+16}{108}.
\end{aligned}$$

□

Lemma 3.14. *For $m >$ odd $n \geq 5$, $\nu(D_{m,n}) \leq \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{m^3}{4}$.*

Proof. By Lemma 3.12, for $s = 1$, we have

$$\begin{aligned}
&\nu(D_{m,n})\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)+2m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-n^2-7m-n+5+(2m\cdot n+4m+13n+8)(\frac{m-1}{n})^2}{12}\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)+2m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-n^2-7m-n+5+(3m\cdot n-m(n-4)+3\times 5n-2(n-4))(\frac{m-1}{n})^2}{12}\\
\leq \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)+2m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-n^2-7m-n+5+(m+3)(m-1)^2}{12}\\
= \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)+3m^3-m\cdot n(m-n)-m^2(n-1)-m\cdot n(m-4)-n^2-12(m-1)-n-4}{12}\\
\leq \quad & \tfrac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \tfrac{m^3}{4}
\end{aligned}$$

for $s \neq 1$, we have

$$
\begin{array}{rl}
& \nu(D_{m,n}) \\
\leq & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{(2n^3-4n^2-8n)\lfloor\frac{m}{n}\rfloor^3-(6m\cdot n^2-3n^3-6m\cdot n-12m+15n)\lfloor\frac{m}{n}\rfloor^2}{12} \\
& + \frac{(6m^2n-6m\cdot n^2+n^3-6m\cdot n+4n^2+24m-7n)\lfloor\frac{m}{n}\rfloor+6m^2-6m}{12} \\
= & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{(2n^3-4n^2-8n)(\frac{m-s}{n})^3-(6m\cdot n^2-3n^3-6m\cdot n-12m+15n)(\frac{m-s}{n})^2}{12} \\
& + \frac{(6m^2n-6m\cdot n^2+n^3-6m\cdot n+4n^2+24m-7n)\frac{m-s}{n}+6m^2-6m}{12} \\
= & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{2m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-13m-2s^3+3n\cdot s^2-n0^2\cdot s+6m\cdot s-4n\cdot s+7s}{12} \\
& + \frac{2m^3\cdot n+4m^3+9m^2\cdot n+6m\cdot n\cdot s-6m\cdot n\cdot s^2-12m\cdot s^2s-15n\cdot s^2+8s^3+4n\cdot s^3}{12n^2} \\
= & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{2m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-13m-2s^3+3n\cdot s^2-n0^2\cdot s+6m\cdot s-4n\cdot s+7s}{12} \\
& + \frac{5m^3\cdot n+6m\cdot n\cdot s-2m^2\cdot n(m-5)-m^3(n-4)-m^2\cdot n-2m\cdot ns^2-4m\cdot s^2-15n\cdot s^2-8(m-s)s^2-4n(m-s)s^2)}{12n^2} \\
= & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{2m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-13m-2s^3+3n\cdot s^2-n0^2\cdot s+6m\cdot s-4n\cdot s+7s+m^3+6m}{12} \\
\leq & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{3m^3-3m^2\cdot n+m\cdot n^2+4m\cdot n-7m-2(n-1)^3+3n\cdot(n-1)^2-n0^2(n-1)+6m(n-1)-4n(n-1)+7(n-1)}{12} \\
= & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{3m^3-3m^2\cdot n+m\cdot n^2+10m\cdot n-3n^2-13m+8n-5}{12} \\
= & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{3m^3-m\cdot n(m-n)-2m\cdot n(m-5)-13m-n^2-2n(n-4)-5}{12} \\
\leq & \frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{m^3}{4}
\end{array}
$$

$\square$

By Lemmas 3.6, 3.7, 3.13 and 3.14, we have

Theorem 3.1. *For $n \geq 3$,*

$$
cr(K_m \times C_n) \quad \leq \begin{cases}
\frac{n\cdot m(m-1)(m-2)(3m-5)}{12} & \text{for } m \geq 4 \text{ and even } n \geq 4, \\
\frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{m(m-1)}{2} & \text{for } 4 \leq m \leq \text{ odd } n, \\
\frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{28m^3-54m^2+42m+16}{108} & \text{for } m > n = 3, \\
\frac{n\cdot m(m-1)(m-2)(3m-5)}{12} + \frac{m^3}{4} & \text{for } m > \text{ odd } n \geq 5.
\end{cases}
$$

4 Lower bounds of $cr(K_m \times P_n)$ and $cr(K_m \times C_n)$

We shall introduce the lower bound method proposed by Leighton [7]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An embedding of G_1 in G_2 is a couple of mapping (φ, κ) satisfying

$$\varphi : V_1 \to V_2 \text{ is an injection}$$

$$\kappa : E_1 \to \{\text{set of all paths in } G_2\},$$

such that if $uv \in E_1$ then $\kappa(uv)$ is a path between $\varphi(u)$ and $\varphi(v)$. For any $e \in E_2$ define

$$cg_e(\varphi, \kappa) = |\{f \in E_1 : e \in \kappa(f)\}|$$

and

$$cg(\varphi, \kappa) = \max_{e \in E_2}\{cg_e(\varphi, \kappa)\}.$$

The value $cg(\varphi, \kappa)$ is called congestion.

Lemma 4.1. *[7] Let (φ,κ) be an embedding of G_1 in G_2 with congestion $cg(\varphi,\kappa)$. Let $\Delta(G_2)$ denote the maximal degree of G_2. Then*

$$cr(G_2) \geq \frac{cr(G_1)}{cg^2(\varphi,\kappa)} - \frac{|V_2|}{2}\Delta^2(G_2).$$

Let $K^x_{m,n}$ be the complete bipartite multigraph of $m+n$ vertices, in which every two vertices are joined by x parallel edges.

According to De Klerk [3] and Kainen [6], the following lemmas hold.

Lemma 4.2. *[3] $cr(K_{m,n}) \geq 0.8594\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$.*

Lemma 4.3. *[6] $cr(K^x_{m,n}) = x^2 cr(K_{m,n})$.*

Now we are in a position to show the lower bound of $cr(K_{m,m}-mK_2)$ and $cr(K_{m,2m}-mK_{1,2}$.

Theorem 4.1. $cr(K_{m,m}-mK_2) \geq \frac{0.8594}{(1+\frac{3}{m-1})^2}\lfloor\frac{m}{2}\rfloor^2\lfloor\frac{m-1}{2}\rfloor^2 - m(m-1)^2$.

Proof. By Lemmas 4.1-4.3, we only need to construct an embedding (φ,κ) of $K^{(m-1)(m-2)}_{m,m}$ into $K_{m,m}-mK_2$ with congestion $cg(\varphi,\kappa)=(m-2)(m+2)$.

Let $\alpha^k_i\beta^k_i$ be the k-th $(m-1,2)$-arrangement, where $\alpha^k_i,\beta^k_i \in \{0,1,2,\cdots,m-1\}-\{i\}$ and $\alpha^k_i \neq \beta^k_i$. Let

$$\begin{aligned}
V(K^{(m-1)(m-2)}_{m,m}) &= \{u_i, v_i \mid 0 \leq i \leq m-1\},\\
E(K^{(m-1)(m-2)}_{m,m}) &= \{(u_i,v_j)^k \mid 0 \leq i,j \leq m-1, 1 \leq k \leq (m-1)(m-2)\},\\
V(K_{m,m}-mK_2) &= \{a_i, b_i \mid 0 \leq i \leq m-1\},\\
E(K_{m,m}-mK_2) &= \{(a_i,b_j) \mid 0 \leq i \neq j \leq m-1\}.
\end{aligned}$$

Let $\varphi(u_i)=a_i$, $\varphi(v_i)=b_i$, $\kappa((u_i,v_i)^k)=P_{a_ib_{\alpha^k_i}a_{\beta^k_i}b_i}$ for $0 \leq i \leq m-1$, and $\kappa((u_i,v_j)^k) = (a_i,b_j)$ for $0 \leq i \neq j \leq m-1$. Then $cg_e(\varphi,\kappa)=(m-2)(m+2)$ for every $e \in E(K^{(m-1)(m-2)}_{m,m})$. This completes the proof of Theorem 4.1. □

Theorem 4.2. $cr(K_{m,2m}-mK_{1,2}) \geq \frac{0.8594}{(1+\frac{3}{m-1})^2}m(m-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor - 6m(m-1)^2$.

Proof. By Lemmas 4.1-4.3, we only need to construct an embedding (φ,κ) of $K^{(m-1)(m-2)}_{m,2m}$ into $K_{m,2m}-mK_{1,2}$ with congestion $cg(\varphi,\kappa)=(m-2)(m+2)$.

Let $\alpha_i^k\beta_i^k$ be the k-th $(m-1,2)$-arrangement, where $\alpha_i^k, \beta_i^k \in \{0,1,2,\cdots,m-1\}-\{i\}$ and $\alpha_i^k \neq \beta_i^k$. Let

$$\begin{aligned}
V(K_{m,2m}^{(m-1)(m-2)}) &= \{u_i, v_i, w_i \mid 0 \le i \le m-1\},\\
E(K_{m,2m}^{(m-1)(m-2)}) &= \{(u_i, v_j)^k, (u_i, w_j)^k \mid 0 \le i, j \le m-1, 1 \le k \le (m-1)(m-2)\},\\
V(K_{m,2m} - mK_{1,2}) &= \{a_i, b_i, c_i \mid 0 \le i \le m-1\},\\
E(K_{m,2m} - mK_{1,2}) &= \{(a_i, b_j), (a_i, c_j) \mid 0 \le i \neq j \le m-1\}.
\end{aligned}$$

Let $\varphi(u_i) = a_i$, $\varphi(v_i) = b_i$, $\varphi(w_i) = c_i$, $\kappa((u_i, v_i)^k) = P_{a_i b_{\alpha_i^k} a_{\beta_i^k} b_i}$, $\kappa((u_i, w_i)^k) = P_{a_i c_{\alpha_i^k} a_{\beta_i^k} c_i}$ for $0 \le i \le m-1$, and $\kappa((u_i, v_j)^k) = (a_i, b_j)$, $\kappa((u_i, w_j)^k) = (a_i, c_j)$ for $0 \le i \neq j \le m-1$. Then $cg_e(\varphi, \kappa) = (m-2)(m+2)$ for every $e \in E(K_{m,2m}^{(m-1)(m-2)})$. This completes the proof of Theorem 4.2. $\square$

Let D_P (D_C) be an arbitrary drawing of $K_m \times P_n$ $(K_m \times C_n)$. By Lemma 1.1, we have $\nu(D_P) \ge \sum_{j=0}^{n-2} \nu_{D_P}(E_j)$ $(\nu(D_C) \ge \sum_{j=0}^{n-1} \nu_{D_C}(E_j))$. Since $(K_m \times P_n)[E^j] \cong (K_m \times C_n)[E^j] \cong K_{m,m} - mK_2$ and $(K_m \times P_n)[E^j \cup E^{j+1}] \cong (K_m \times C_n)[E^j \cup E^{j+1}] \cong K_{m,2m} - mK_{1,2}$, where $G[X]$ denotes the subgraph of G induced by $X \subseteq E(G)$, by Theorems 4.1 and 4.2, we have

Theorem 4.3.

$$cr(K_m \times P_n) \ge \begin{cases} \frac{n-2}{2}\left(\frac{0.8594}{(1+\frac{3}{m-1})^2} m(m-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor - 6m(m-1)^2\right) & \\ +\left(\frac{0.8594}{(1+\frac{3}{m-1})^2}\lfloor\frac{m}{2}\rfloor^2\lfloor\frac{m-1}{2}\rfloor^2 - m(m-1)^2\right) & \textit{for even } n \\ \frac{n-1}{2}\left(\frac{0.8594}{(1+\frac{3}{m-1})^2} m(m-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor - 6m(m-1)^2\right) & \textit{for odd } n. \end{cases}$$

Theorem 4.4.

$$cr(K_m \times C_n) \ge \begin{cases} \frac{n-1}{2}\left(\frac{0.8594}{(1+\frac{3}{m-1})^2} m(m-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor - 6m(m-1)^2\right) & \\ +\left(\frac{0.8594}{(1+\frac{3}{m-1})^2}\lfloor\frac{m}{2}\rfloor^2\lfloor\frac{m-1}{2}\rfloor^2 - m(m-1)^2\right) & \textit{for odd } n \\ \frac{n}{2}\left(\frac{0.8594}{(1+\frac{3}{m-1})^2} m(m-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor - 6m(m-1)^2\right) & \textit{for even } n. \end{cases}$$

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